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ICAPS-08 Tutorial on

External-Memory Graph Search

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External-Memory Graph Search

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Motivation: Recent Successes in Search

Optimal solutions the RUBIK'S CUBE, the (n^2-1)-PUZZLE, and the TOWERS-OF-HANOI problem, all with state spaces of about or more than a quintillion (a billion times a billion) states.

When processing a million states per second, looking at all states corresponds to hundreds of thousands of years.

Even with search heuristics, time and space remain crucial resources: in extreme cases, weeks of computation time, gigabytes of main memory and terabytes of hard disk space have been invested to solve search challenges.

Recent Examples Disk-based Search

RUBIK'S CUBE: 43,252,003,274,489,856,000 states,

2007: By performing a breadth-first search over subsets of configurations, starting with the solved one, in 63 hours with the help of 128 processor cores and 7 terabytes of disk space it was shown that 26 moves suffice.

Recent Examples of Disk-based Search

With recent search enhancements, the average solution time for optimally solving the FIFTEEN-PUZZLE with over 10^13 states is about milliseconds, looking at thousands of states.

The state space of the FIFTEEN-PUZZLE has been completely generated in 3 weeks using 1.4 terabytes of secondary memory.

Tight bounds on the optimal solutions for the THIRTY-FIVE-PUZZLE with over 10^41 states have been computed in more than one month total time using 16 gigabytes RAM and 3 terabytes hard disk.

Recent Examples of Disk-based Search

The 4-peg 30-disk TOWERS-OF-HANOI problem spans a state space of $4^{30} = 1,152,921,504,606,846,976$ states

Optimally solved by integrating a significant number of research results consuming about 400 gigabytes hard disk space in 17 days.

Further Examples

Spin (model checker, ACM distinguished software award)

Divine (state-of-the-art parallel model checker)

Uppaal-CORA (most widely used real-time model checker)

STEAM (c/c++ program validation tool)

Metric FF (best-known metric planner)

IDDP (state-of-the-art optimal multi sequence alignment)

Largest state space: ~3 terabytes disk space, while using only 3.5 gigabytes of RAM taking 20 days (parallel version on 4 processors with shared NFS hard disk took 8 days)

Outline

I. Review of basic graph-search techniques
- limited-memory graph search
- including frontier search

II. Introduction to External-Memory Algorithms and I/O Complexity Analysis

III. External-Memory Search Algorithms

IV. Exploiting Problem Graph Structure

V. Parallel External-Memory Graph Search

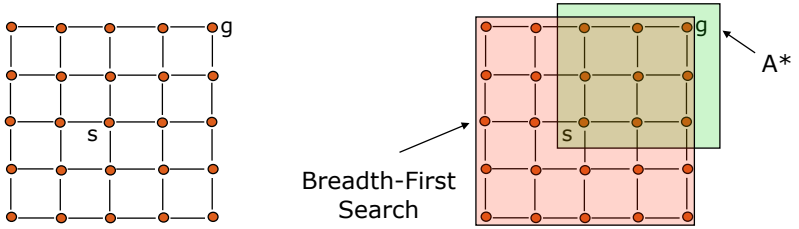
VI. Applications: Model Checking

VII. Advanced Topics: Probabilistic, Semi-Externality Flash-Memory, etc.

Notations

- G: Graph
- V: Set of nodes of G
- E: Set of edges of G
- w: E $\rightarrow$ IR: Weight function that assigns a cost to each edge.
- δ: shortest path distance between two nodes.
- Open list: Search frontier – waiting to be expanded.
- Closed list: Expanded nodes.

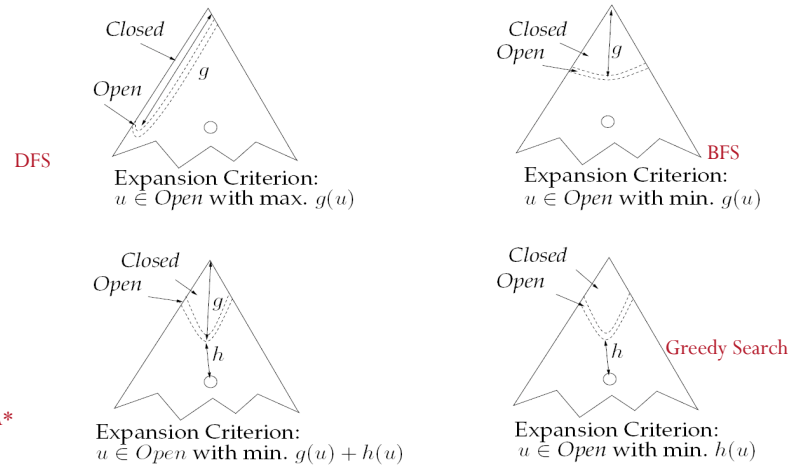
Heuristic Search – A* algorithm



A heuristic estimate is used to guide the search.

- E.g. Straight line distance from the current node to the goal in case of a graph with a geometric layout.

Comparison of Search Algorithms



Heuristics

Admissible Heuristics

- Never over-estimates the optimal path.

➔ Guarantees the optimal path when A* expands a goal node

Consistent Heuristics

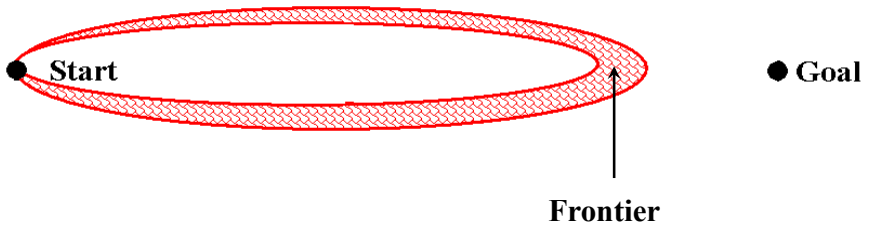
- Never drops fasters than the edge weight.

➔ Guarantees A* never re-opens a node, and is optimally efficient

Divide-and-conquer frontier A*

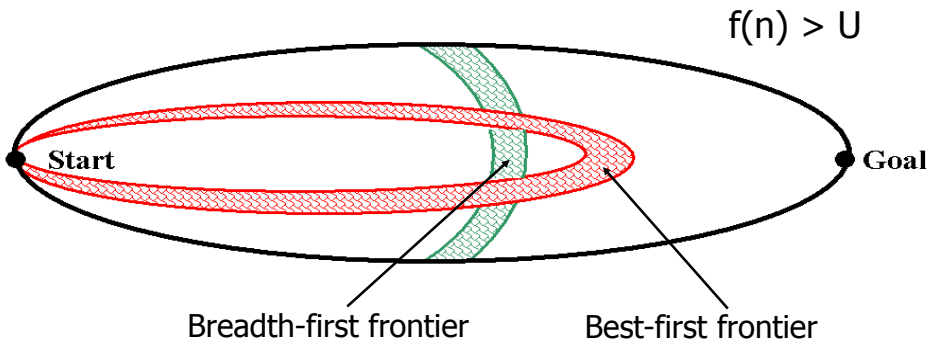
[Korf & Zhang AAAI-00]

- Stores Open list, but not Closed list
- Reconstructs solution using divide-and-conquer method



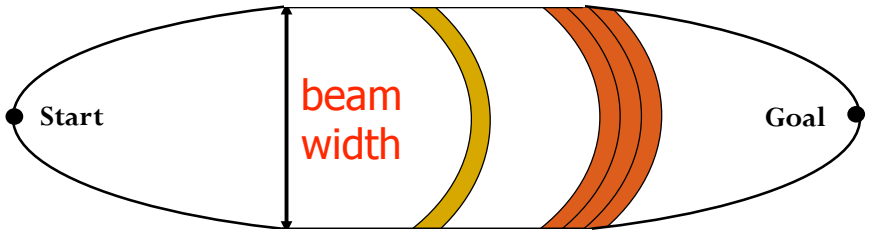
Breadth-first heuristic search [Zhou & Hansen AIJ-06]

Breadth-first branch-and-bound is more memory-efficient than best-first search



Divide-and-conquer beam search

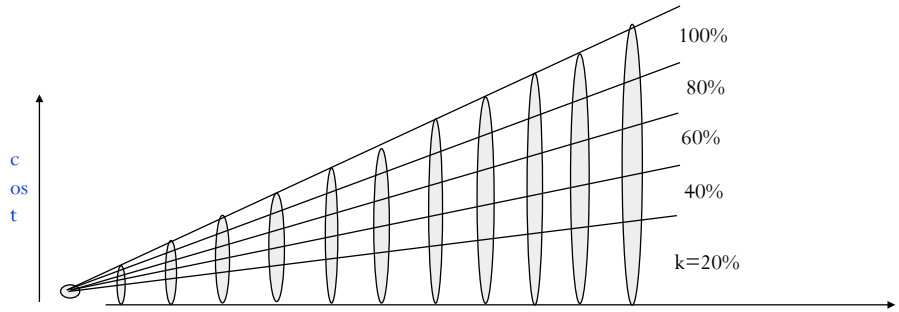
- Stores 3 layers for duplicate elimination and 1 "middle" layer for solution reconstruction
- Uses beam width to limit size of a layer



Divide-and-conquer beam-stack search

- Memory use bounded by 4 × beam width
- Use beam stack to backtrack to a set of nodes
- Allows much wider beam width, which reduces backtracking
- Contains both breadth-first and depth-first branch-and-bound search as special cases

Iterative Broadening Breadth-First Branch-and-Bound

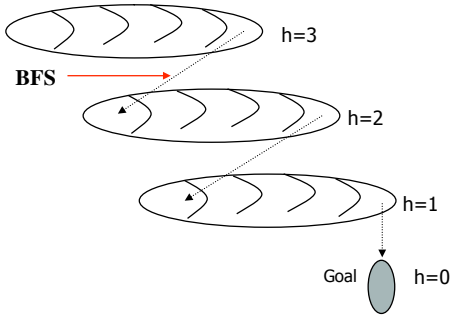


Search frontier

Only pick best k% nodes for expansion.

Enforced Hill-Climbing

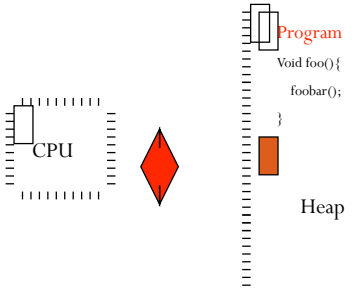
- Most successful planning algorithm



Introduction to EM Algorithms

- Von Neumann RAM Model
- Virtual Memory
- External-Memory Model
- Basic I/O complexity analysis
  - External Scanning
  - External Sorting
- Breadth-First Search
- Graphs
  - Explicit Graphs
  - Implicit Graphs

Von Neumann RAM Model?



- **Main assumptions:**
  - Program and heap fit into the main memory.
  - CPU has a fast constant-time access to the memory contents.

Virtual Memory Management Scheme

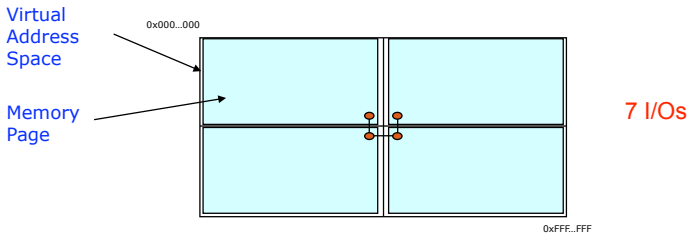
- Address space is divided into memory pages.
- A large virtual address space is mapped to a smaller physical address space.
- If a required address is not in the main memory, a page-fault is triggered.
  - A memory page is moved back from RAM to the hard disk to make space,
  - The required page is loaded from hard disk to RAM.

Virtual Memory

+ works well when word processing, spreadsheet, etc. are used.

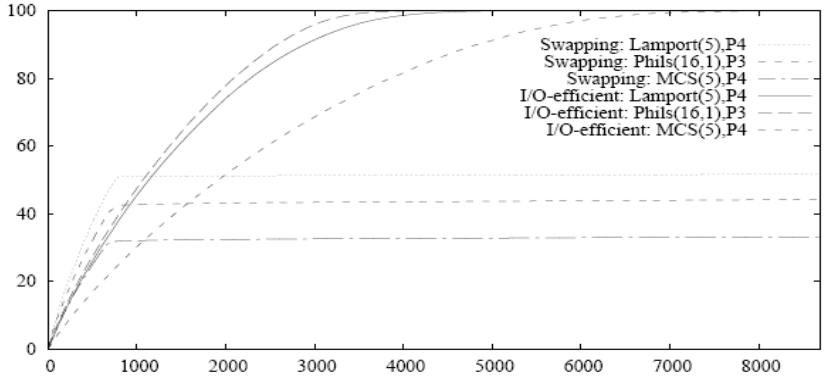
– does not know any thing about the data accesses in an algorithm.

In the worst-case, can result in one page fault for every state access!



7 I/Os

EM better than IM Graph Search?

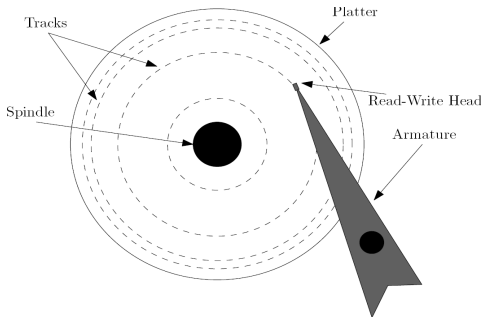


Memory Hierarchy

Latency times		Typical capcity
~2 ns	Registers (x86_64)	16 x 64 bits
3.0 ns	L1 Cache	64 KB
17 ns	L2 Cache	512 KB
23 ns	L3 Cache	2 – 4 MB
86 ns	RAM	4 GB
4.2 ms	Hard disk (7200 rpm)	600 GB

8.3 ms full rotation

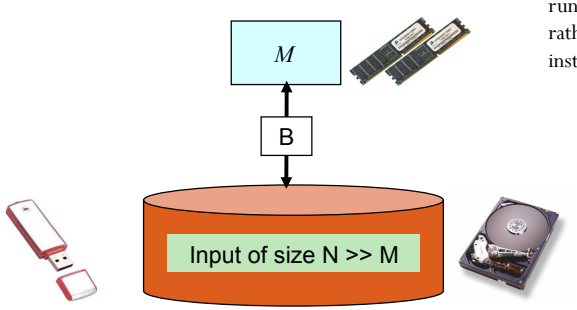
Working of a hard disk



- Data is written on tracks in form of sectors.
- While reading, armature is moved to the desired track.
- Platter is rotated to bring the sector directly under the head.
- A large block of data is read in one rotation.

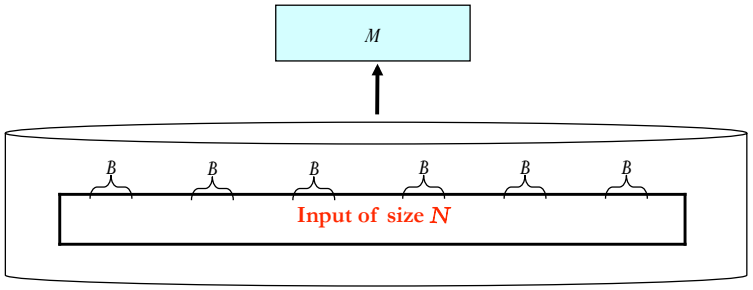
External Memory Model [Aggarwal and Vitter]

If the input size is very large, running time depends on the I/Os rather than on the number of instructions.



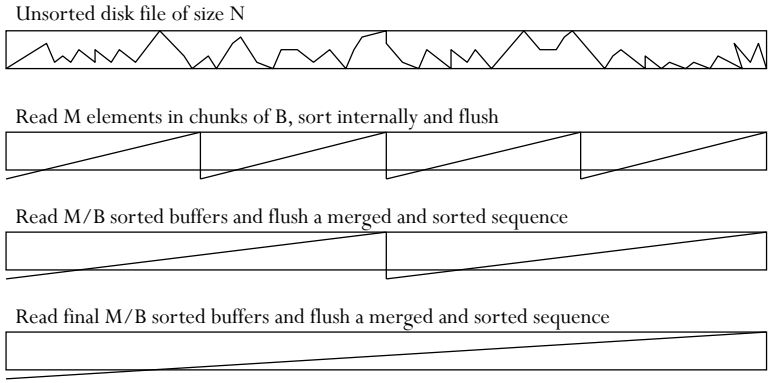
External Scanning

- Given an input of size N, consecutively read B elements in the RAM.



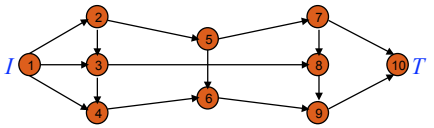
- **I/O complexity:** $scan(N) = \frac{N}{B}$

External Sorting



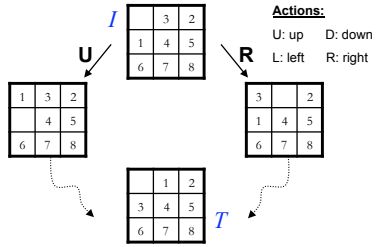
- **I/O complexity:** $sort(N) = O\left(\frac{N}{B}\log_{\frac{M}{B}}\frac{N}{B}\right)$

Explicit vs. Implicit



A path search problem in explicit graphs:

Given a graph, does a path between two nodes I and T exist?



Reachability analysis in implicit graphs:

Given an initial state(s) I and a set of transformation rules, is a desired state T reachable?

Traverse/Generate the graph until T is reached.

Search Algorithms: *DFS, BFS, Dijkstra, A*, etc.)*

What if the graph is too big to fit in the RAM?

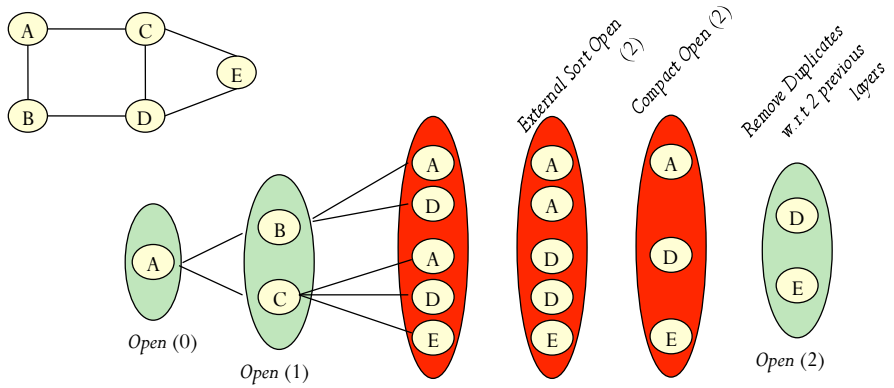
8-puzzle has 9!/2 states … 15-puzzle has **16!/2 ≈ 10 461 394 900 000** states

External-Memory Graph Search

- External BFS
- Delayed Duplicate Detection
- Locality
- External A*
  - Bucket Data Structure
  - I/O Complexity Analysis

External Breadth-First Search

(Munagala and Ranade, SODA'99)



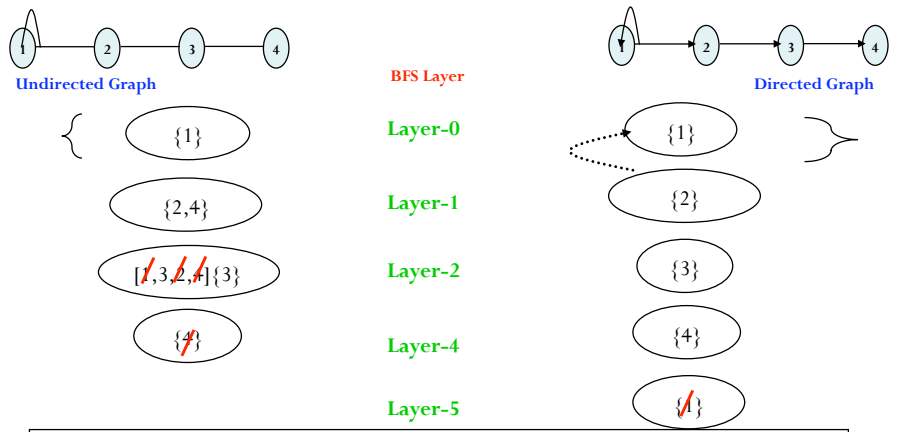
I/O Complexity Analysis of EM-BFS for Explicit Graphs

- Expansion:
  - Sorting the adjacency lists: O(Sort(|V|))
  - Reading the adjacency list of all the nodes: O(|V|)
- Duplicates Removal:
  - Phase I: External sorting followed by scannig. O(Sort(|E|) + Scan(|E|))
  - Phase II: Subtraction of previous two layers: O(Scan(|E|) + Scan(|V|))
- Total: O(|V| +Sort(|E| + |V|)) I/Os

Delayed Duplicate Detection (Korf 2003)

- Essentially idea of Munagala and Ranade applied to implicit graphs …
- Complexity:
  - Phase I: External sorting followed by scannig. O(Sort(|E|) + Scan(|E|))
  - Phase II: Subtraction of previous two layers: O(Scan(|E|) + Scan(|V|))
- Total: O(Sort(|E|) + Scan (|V|)) I/Os

Locality in Breadth-First Search



[Zhou & Hansen 05]

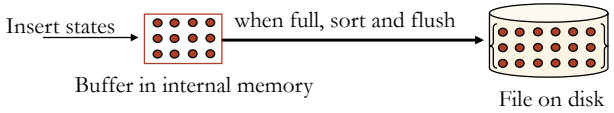
Problems with A* Algorithm

- A* needs to store all the states during exploration.
- A* generates large amount of duplicates that can be removed using an internal hash table – only if it can fit in the main memory.
- A* do not exhibit any locality of expansion. For large state spaces, standard virtual memory management can result in excessive page faults.

Can we follow the strict order of expanding with respect to the minimum g+h value? - Without compromising the optimality?

Data Structure: Bucket

- A Bucket is a set of states, residing on the disk, having the same (g, h) value, where:
  - g = number of transitions needed to transform the initial state to the states of the bucket,
  - and h = Estimated distance of the bucket's state to the goal
- No state is inserted again in a bucket that is expanded.
- If *Active* (being read or written), represented internally by a small buffer.



External A* (E., Jabbar and Schrödl 2004) based on Set A* (Jensen, Veloso, Bryant 2002) and BDDA* (Raffel, E., 1998)

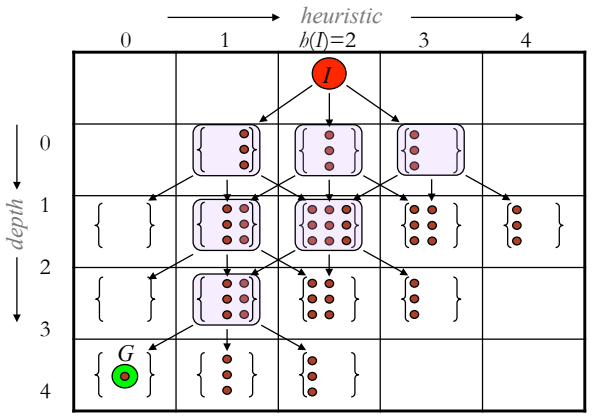
Simulates a priority queue by exploiting the properties of the heuristic function:

- h is a total function!!
- Consistent heuristic estimates.

$\Rightarrow \Delta h = \{-1,0,1,\ldots\}$

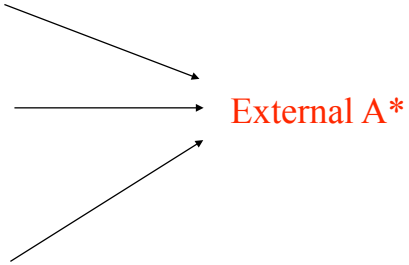
$w'(u,v) = w(u,v) - h(u) + h(v)$

$=> w'(u,v) = 1 + \{-1,0,1\}$



External A*

- Buckets represent temporal locality – cache efficient order of expansion.
- If we store the states in the same bucket together we can exploit the spatial locality.
- Munagala and Ranade's BFS and Korf's delayed duplicate detection for implicit graphs.



Procedure *External A**

$Bucket(0, h(\boldsymbol{I})) \leftarrow \{\boldsymbol{I}\}$

$f_{min} \leftarrow h(\boldsymbol{I})$

while $(f_{min} \neq \infty)$

$g \leftarrow \min\{i \mid Bucket(i, f_{min} - i) \neq \phi\}$

while $(g_{min} \leq f_{min})$

$h \leftarrow f_{min} - g$

$Bucket(g, h) \leftarrow$ *remove duplicates from* $Bucket(g, h)$

$Bucket(g, h) \leftarrow Bucket(g, h) \setminus (Bucket(g-1, h) \cup Bucket(g-2, h))$ // *Subtraction*

$A(f_{min}), A(f_{min}+1), A(f_{min}+2) \leftarrow N(Bucket(g, h))$ // *Neighbours*

$Bucket(g+1, h+1) \leftarrow A(f_{min}+2)$

$Bucket(g+1, h) \leftarrow A(f_{min}+1) \cup Bucket(g+1, h)$

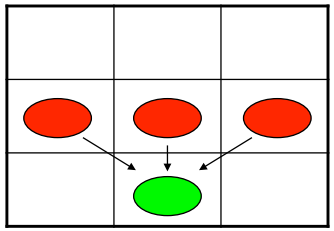
$Bucket(g+1, h-1) \leftarrow A(f_{min}) \cup Bucket(g+1, h-1)$

$g \leftarrow g + 1$

$f_{min} \leftarrow \min\{i + j > f_{min} \mid Bucket(i, j) \neq \phi\} \cup \{\infty\}$

I/O Complexity Analysis

- Internal A* => Each edge is looked at most once.
- Duplicates Removal:
  - Sorting the green bucket having one state for every edge from the 3 red buckets.
  - Scanning and compaction.
    - $O(sort(|E|))$



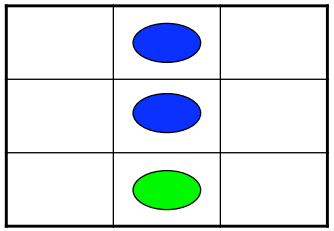
Total I/O complexity:

$\theta(sort(|E|) + scan(|V|))$ I/Os

Cache-Efficient at all levels!!!

Complexity Analysis

- Subtraction:
  - Removing states of blue buckets (duplicates free) from the green one.
    - $O(scan(|V|) + scan(|E|))$



Total I/O complexity:

$\theta(sort(|E|) + scan(|V|))$ I/Os

Cache-Efficient at all levels!!!

I/O Performance of External A*

Theorem: The complexity of External A* in an implicit unweighted and undirected graph with a consistent heuristic estimate is bounded by

$$O(\text{sort}(|E|) + \text{scan}(|V|)) \text{ I/Os.}$$

Test Run – Generated states

g/h	1	2	3	4	5	6	7	8	9	10	11
0	-	-	-	1+0	-	-	-	-	-	-	-
1	-	-	-	-	2+0	-	-	-	-	-	-
2	-	-	-	0+4	-	2+0	-	-	-	-	-
3	-	-	-	-	7+3	-	4+0	-	-	-	-
4	-	-	-	0+7	-	13+4	-	10+0	-	-	-
5	-	-	-	-	5+15	-	24+10	-	24+0	-	-
6	-	-	-	0+6	-	12+26	-	46+28	-	44+0	-
7	-	-	-	-	9+10	-	20+51	-	99+57	-	76+0
8	-	-	-	0+8	-	15+25	-	48+137	-	195+0	-
9	-	-	-	-	4+17	-	45+52	-	203+0	-	-
10	-	-	-	0+3	-	13+49	-	92+0	-	-	-
11	-	-	-	-	2+19	-	46+0	-	-	-	-
12	-	-	-	0+5	-	31+0	-	-	-	-	-
13	-	-	0+2	-	10+0	-	-	-	-	-	-
14	-	0+2	-	5+0	-	-	-	-	-	-	-
15	0+2	-	5+0	-	-	-	-	-	-	-	-

Test Run - Duplicates

g/h	0	1	2	3	4	5	6	7	8	9	10	11
0	-	-	-	-	-	-	-	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	1+1	-	-	-	-	-	-	-
3	-	-	-	-	-	2+2	-	-	-	-	-	-
4	-	-	-	-	3+2	-	3+2	-	-	-	-	-
5	-	-	-	-	-	8+6	-	6+4	-	-	-	-
6	-	-	-	-	1+2	-	16+12	-	14+10	-	-	-
7	-	-	-	-	-	6+6	-	24+24	-	26+24	-	-
8	-	-	-	-	3+10	-	10+10	-	52+50	-	-	-
9	-	-	-	-	-	9+7	-	29+23	-	-	-	-
10	-	-	-	-	0+2	-	21+20	-	-	-	-	-
11	-	-	-	-	-	6+5	-	-	-	-	-	-
12	-	-	-	-	0+1	-	-	-	-	-	-	-
13	-	-	-	-	-	-	-	-	-	-	-	-
14	-	-	-	-	-	-	-	-	-	-	-	-
15	-	1+0	-	-	-	-	-	-	-	-	-	-

Exploiting Problem Graph Structure

Hash-based Duplicate Detection

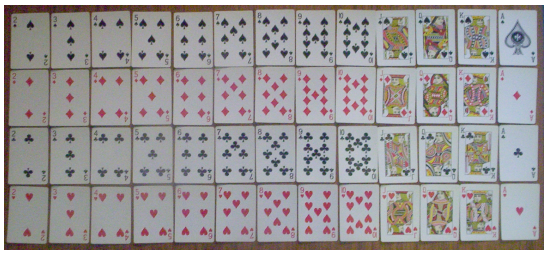
- Basic principle

Structured Duplicate Detection

- Basic principles
- Manual and automated abstraction
- Edge partitioning
- External memory pattern databases

Hash-based Duplicate Detection

1. Hash cards by suit
2. Sort suit internally



Essentials of hash-based DDD

Two orthogonal hash functions one for external and one for internal duplicate elimination

Read one file (partitioned by first hash) at a time into RAM, merge duplicates (using second hash) and flush file

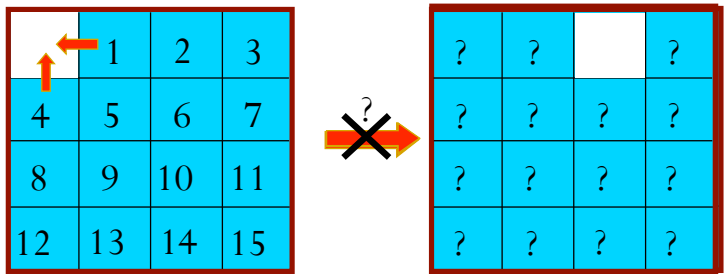
Examples:

15-puzzle: one based on first row, one based on last three row

Tower of Hanoi: one based on the largest discs, one based on the smallest discs

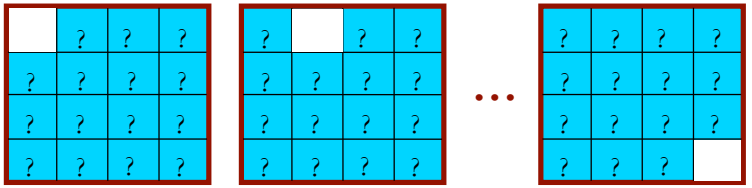
Structured Duplicate Detection

- Idea: localize memory references in duplicate detection by exploiting graph structure
- Example: Fifteen-puzzle



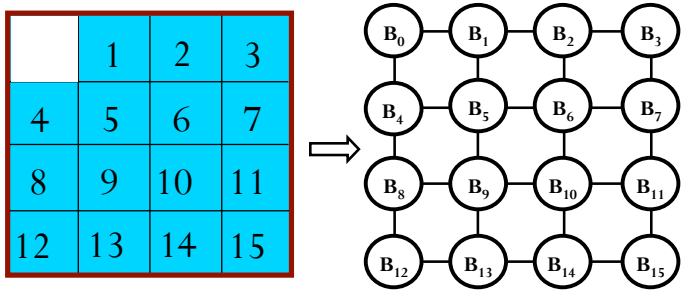
State-space projection function

- Many-to-one mapping from original state space to abstract state space
- Created by ignoring some state variables
- Example



Abstract state-space graph

- Created by state-space projection function
- Example

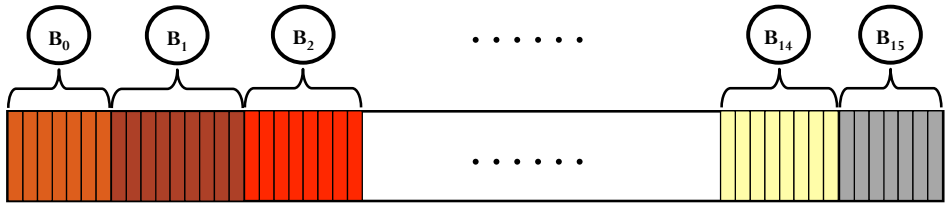


> 10 trillion states

16 abstract states

Partition stored nodes

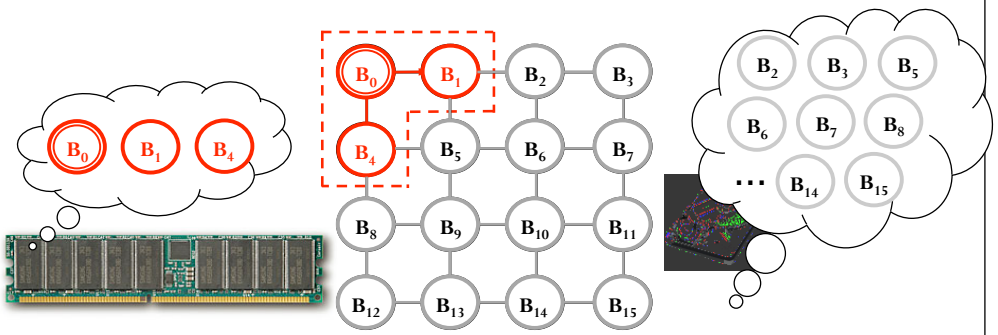
Open and Closed lists are partitioned into blocks of nodes, with one block for each abstract node in abstract state-space graph



Logical memory

Duplicate-detection scope

A set of blocks (of stored nodes) that is guaranteed to contain all stored successor nodes of the currently-expanding node



When is disk I/O needed?

- If internal memory is full, write blocks outside current duplicate-detection scope to disk
- If any blocks in current duplicate-detection scope are not in memory, read missing blocks from disk

How to reduce disk I/O?

Given a set of nodes on search frontier, expand nodes in an order such that

- Nodes in the same duplicate-detection scope are expanded together
- Nodes in duplicate-detection scopes with overlapping abstract nodes, are expanded near each other

Locality-preserving abstraction

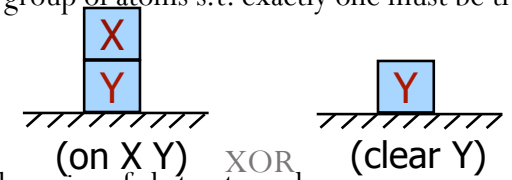
- Max. duplicate-detection scope ratio δ

$$\delta = \frac{\text{max \# of successors of an abstract state}}{\text{size of abstract graph}}$$

  - Measures degree of graph local structure
  - ≈ % of nodes that must be stored in RAM
  - Smaller δ → Less RAM needed
- Search for abstraction that minimizes δ

Exploiting state constraints

- XOR group: a group of atoms s.t. exactly one must be true at any time



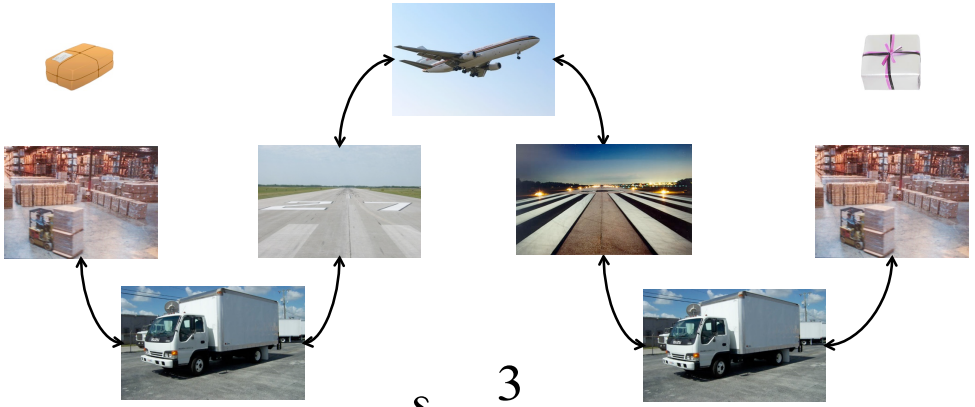
XOR

- Advantage: reduce size of abstract graph
- Example: 2 XOR groups of 5 atoms each size of abstract graph =
  - $2^{5+5} = 1024$ ← without XOR constraints
  - 5 X 5 = 25 ← with XOR constraints

Greedy abstraction algorithm

- Starts with empty set of abstraction atoms
- Mark all XOR groups as unselected
- While (size of abstract graph ≤ M)
  - Find an unselected XOR group P_i s.t. union of abstraction atoms and P_i creates abstract graph with minimum δ
  - Add P_i into set of abstraction atoms
  - Mark P_i as selected

Example: Logistics

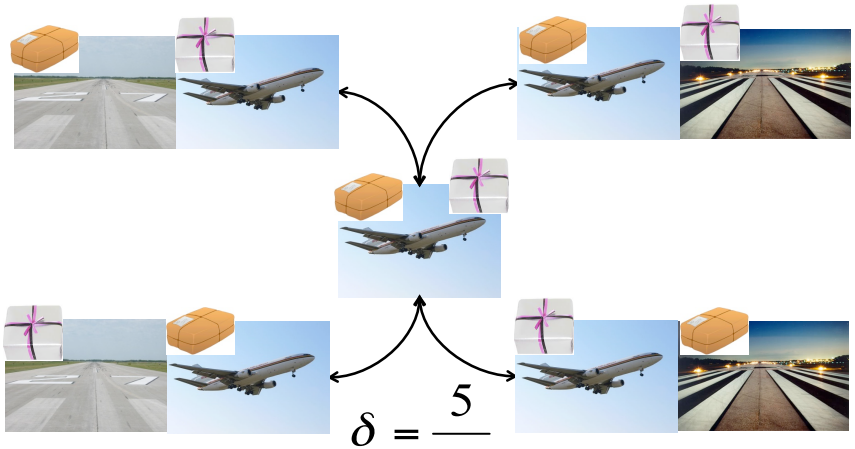


$$\delta = \frac{3}{7}$$

Abstraction based on truck locations

$$\delta = 1$$

Largest duplicate-detection scope based on locations of 2 packages

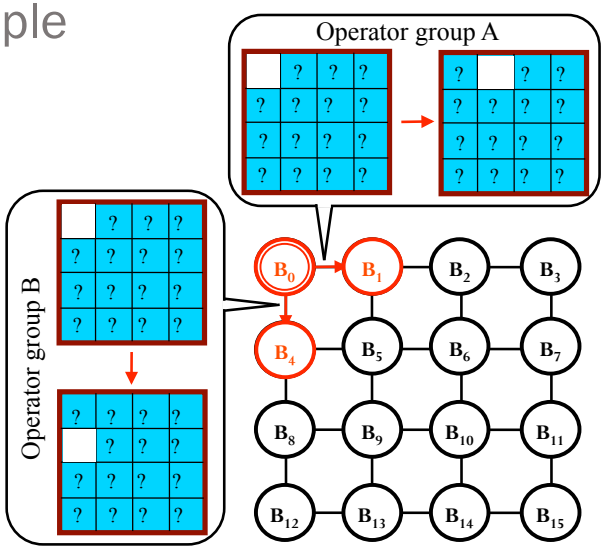


$$\delta = \frac{5}{49}$$

Operator grouping

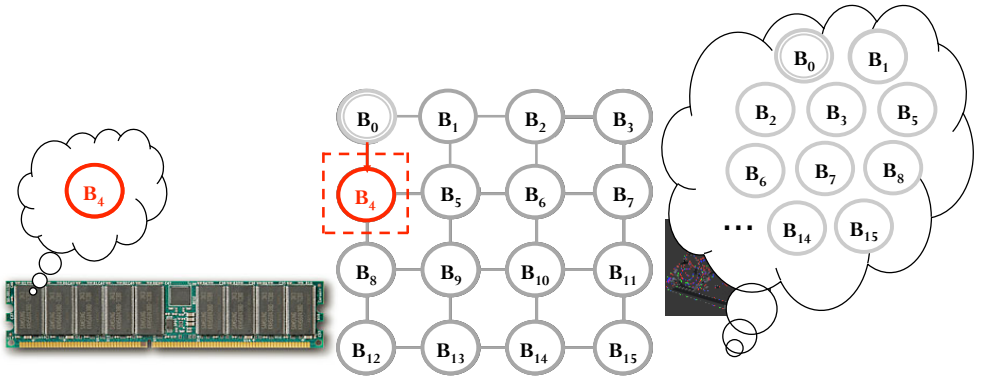
- Exploits structure in operator space
- Divides operators into operator groups for each abstract state
- Operators belong to the same group if they
  - are applicable to the same abstract state
  - lead to the same successor abstract state

Example



Edge Partitioning

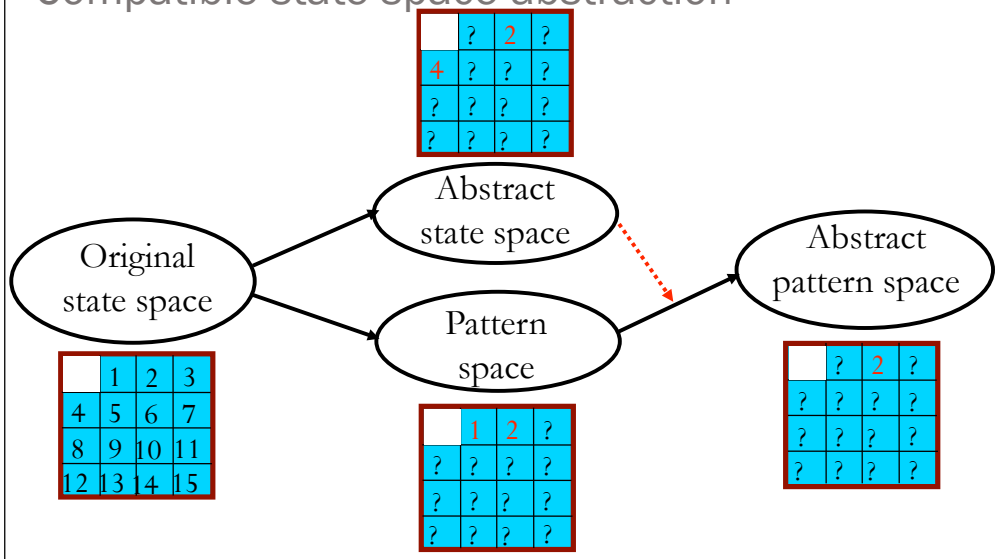
Reduces duplicate-detection scope to one block of stored nodes – Guaranteed!



External-memory pattern database

- Creating an external-memory PDB
  - Breadth-first search in pattern space using delayed or structured duplicate detection
- Two ways of using an external-memory PDB
  - Compress PDB to fit in RAM
  - Use structured duplicate detection to localize references to PDB, so only a small fraction of PDB needs to be stored in RAM at a time

Compatible state-space abstraction



Parallel External-Memory Graph Search

- Motivation Shared and Distributed Environments
- Parallel Delayed Duplicate Detection
  - Parallel Expansion
  - Distributed Sorting
- Parallel Structured Duplicate Detection
  - Finding Disjoint Duplicate Detection Scopes
  - Locking

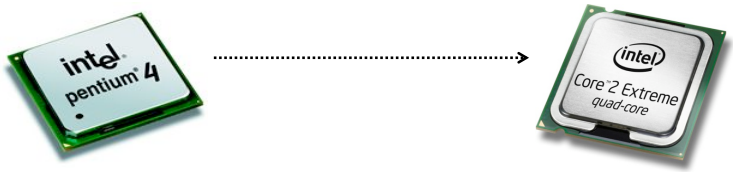
Motivation

Parallel and External Memory Graph Search Synergies:

- They need partitioned access to large sets of data
- This data needs to be processed individually.
- Limited information transfer between two partitions
- Streaming in external memory programs relates to Communication Queues in distributed programs

  (as communication often realized on files)
- Good external implementations often lead to good parallel implementations

Parallel Shared Memory Graph Search

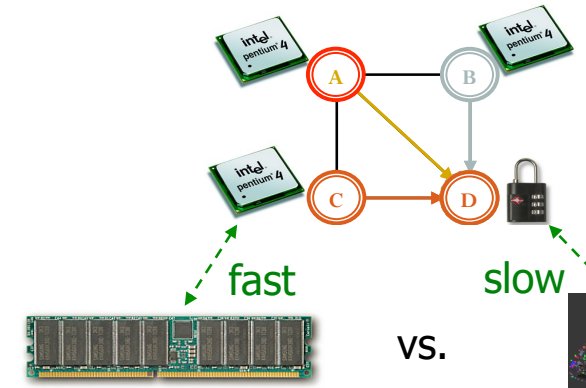


Single-core CPU Multi-core CPU

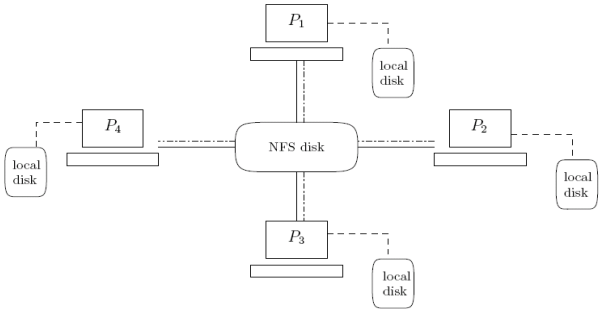
- Parallelization is important for multi-core CPUs
- But parallelizing graph-search algorithms such as breadth-first search, Dijkstra's algorithm, and A* is challenging
- Issues: Load balancing, Locking, and most importantly …

Bottleneck: Duplicate detection

- Duplicate paths cause parallelization overhead



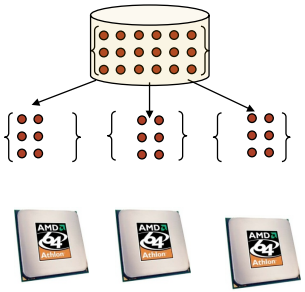
Distributed Search over the Network



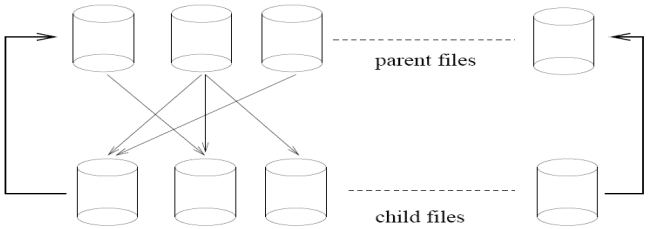
- Distributed setting provides more space.
- Experiments show that internal time dominates I/O.

Exploiting Independence

- Since each state in a Bucket is independent of the other – they can be expanded in parallel.
- Duplicates removal can be distributed on different processors.
- Bulk (Streamed) transfers much better than single ones.

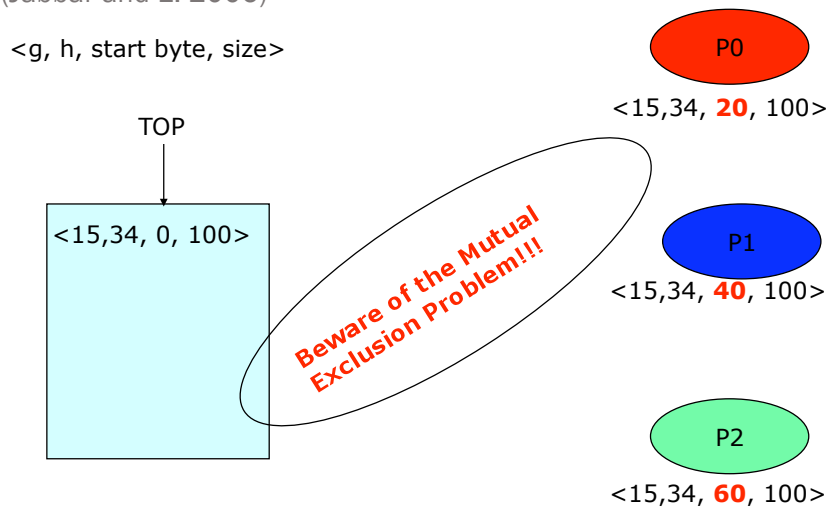


Parallel Breadth-First Frontier Search Enumerating 15-Puzzle (Korf and Schultze 2006)



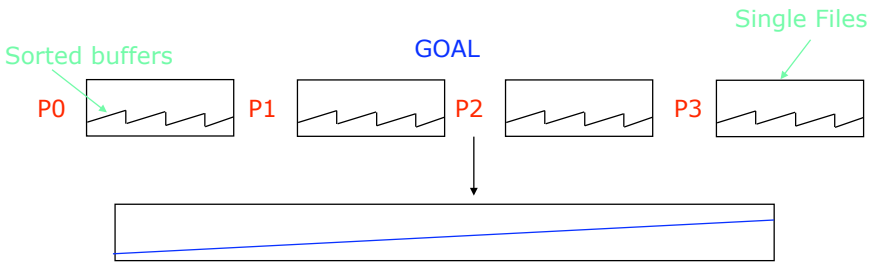
- Hash function partitions both layers into files.
- If a layer is done, children files are renamed into parent files.
- For parallel processing a work queue contains parent files waiting to be expanded, and child files waiting to be merged

Distributed Queue for Parallel Best-First Search (Jabbar and E. 2006)



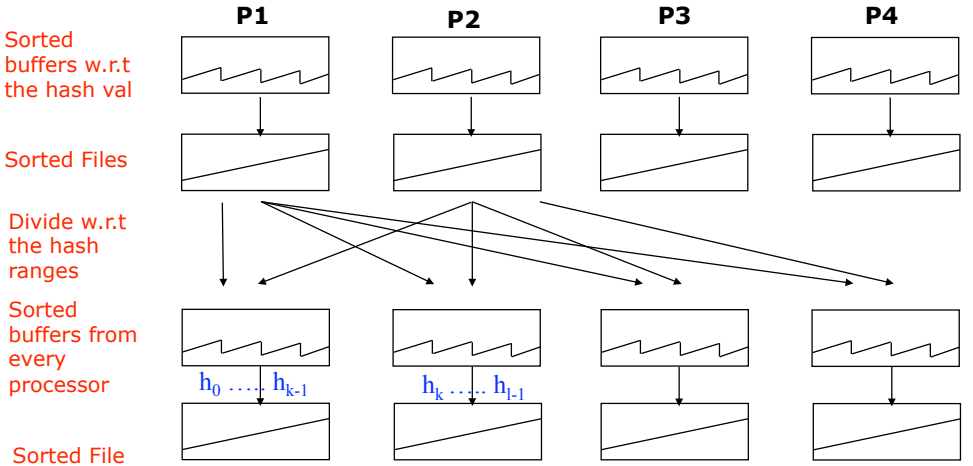
Distributed Delayed Duplicate Detection

- Each state can appear several times in a bucket.
- A bucket has to be searched completely for the duplicates.



Problem: Concurrent Writes !!!!

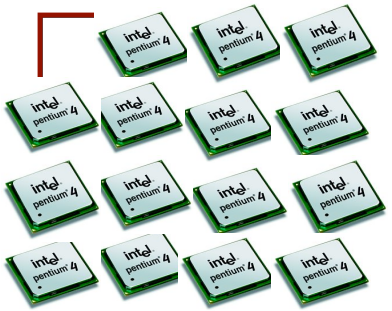
Multiple Processors - Multiple Disks Variant



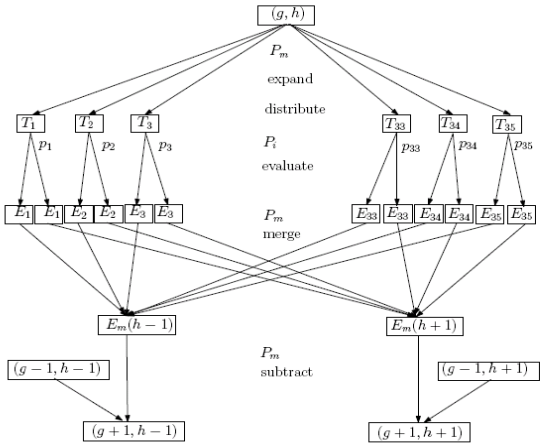
Distributed Heuristic Evaluation (E., Jabbar, Kissmann 2008)

- Assume one child processor for each tile one master processor

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15



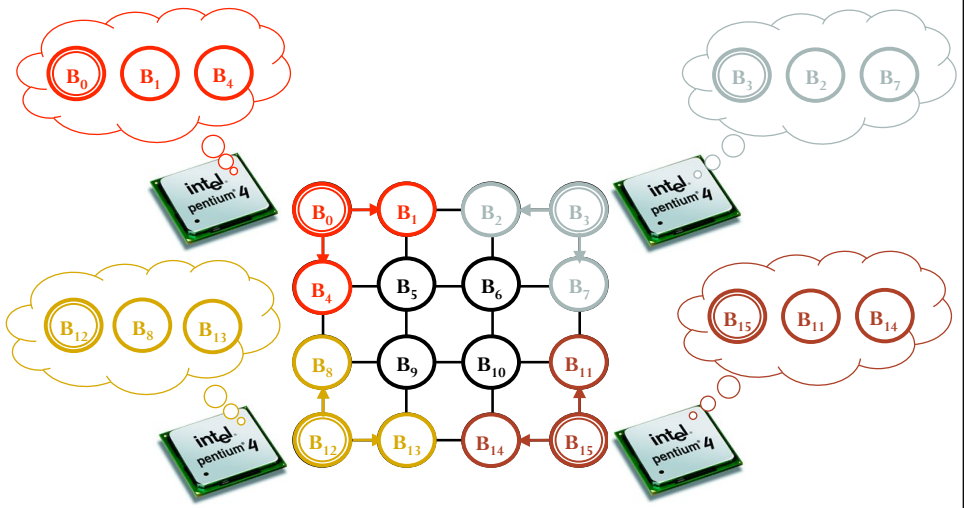
Distributed Heuristic Evaluation



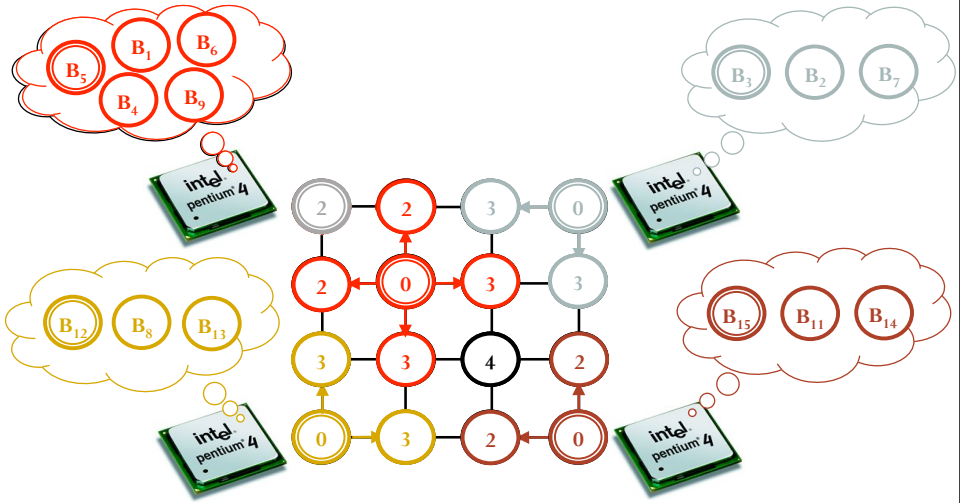
Distributed Pattern Database Search

- Only pattern databases that include the client tile need to be loaded on the client
- Because multiple tiles in pattern, the PDB is loaded on multiple machines
- In 15-Puzzle with corner and fringe PDB this saves RAM in the order of factor 2 on each machine, compared to loading all
- In 36-Puzzle with 6-tile pattern databases this saves RAM in the order of factor 6 on each machine, compared to loading all
- Extends to additive pattern databases

Disjoint duplicate-detection scopes

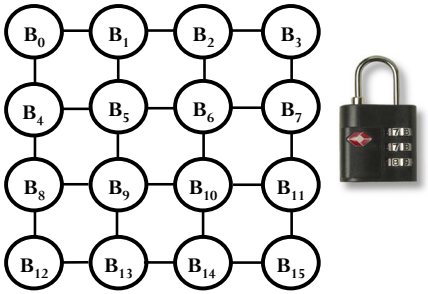


Finding disjoint duplicate-detection scopes



Implementation of Parallel SDD

Only needs a single mutex lock



System Verification

- Need for Verification of System
- (Automata-based) Model Checking
- Safety and Liveness
- External LTL Model Checking
- Alternative Approaches

Motivation

- **Software MUST be correct!**

Ariane-5 rocket destruction

Cost: €500 Million

Cause: over-flow while converting 64-bit to 16-bit integer

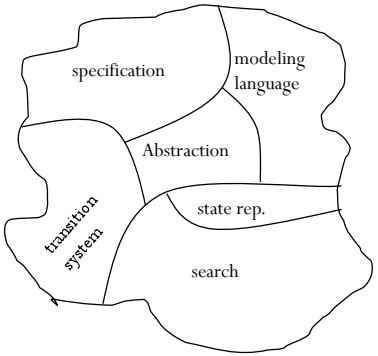
Failed interception of Scud by Patriot missile

Cost: 28 dead 100 injured

Cause: precision lost in using 24-bit registers while calculating `1/10 x elapsed time`.

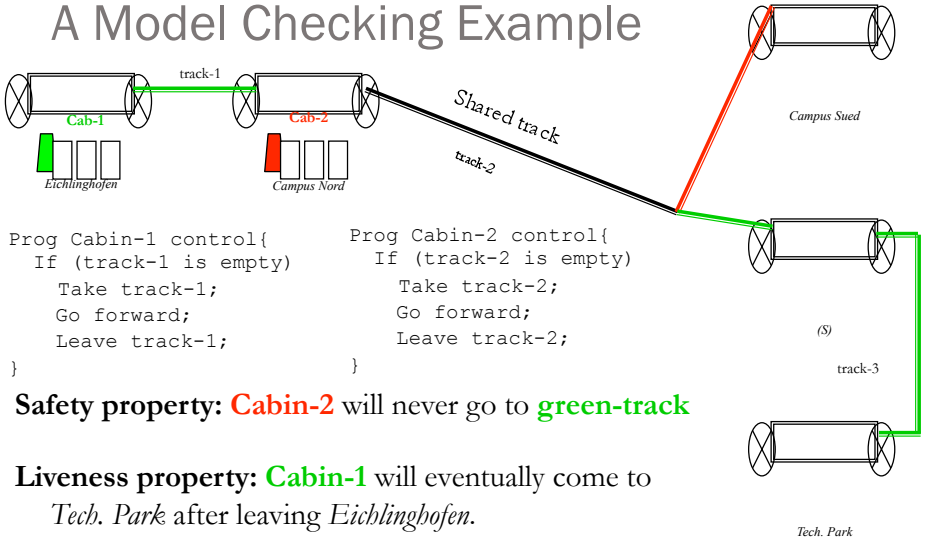
Model Checking

- **Given**
  - A model of a system.
  - A specification property
- **Model Checking Problem:** Does the system satisfy the property ?
- An exhausting exploration of the state space.
- **Problem:** How to cope with large state spaces that do not fit into the main memory?
- **In Practice:** successes in finding bugs.



A Typical Model Checker

A Model Checking Example



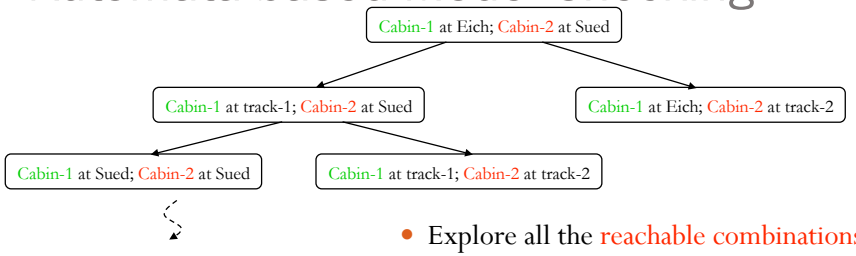

```
Prog Cabin-1 control{
  If (track-1 is empty)
    Take track-1;
    Go forward;
    Leave track-1;
}
```

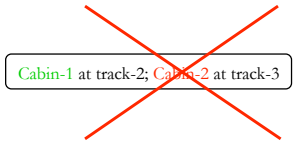
```
Prog Cabin-2 control{
  If (track-2 is empty)
    Take track-2;
    Go forward;
    Leave track-2;
}
```

Safety property: **Cabin-2** will never go to **green-track**

Liveness property: **Cabin-1** will eventually come to *Tech. Park* after leaving *Eichlinghofen*.

Automata-based Model Checking

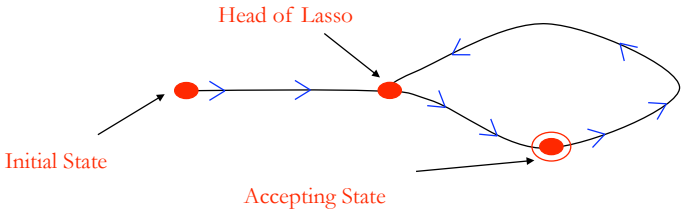




- Explore all the reachable combinations of the system.
- Search for the state where a property is NOT satisfied.
- Return a **counter-example**.
- **Problem:** 100 tracks and 10 cabins – Cabin-1 can be at 100 tracks, Cabin-2 at 99 tracks ≈ **1 731 030 950 000 states**

Liveness Property

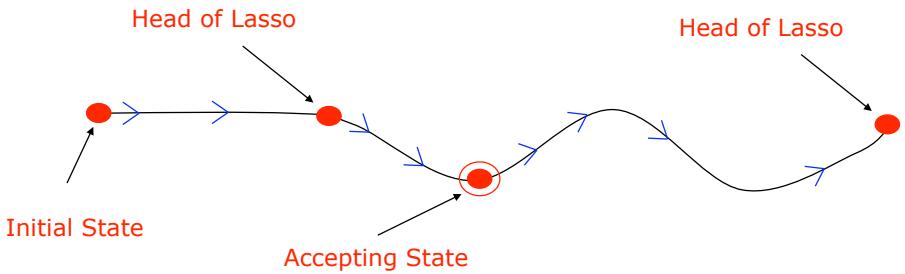
- Product of Automata for System and for the Property
- Search for a path that visits an accepting state infinitely often.
- Nested Depth-first search look for a state that is already residing on the stack.



DFS does not show any locality of reference => Not Suitable for External Search!

Liveness as Safety (Schuppan and Biere, 2005)

- Explicitly unroll the lasso.
- Search for the head again.

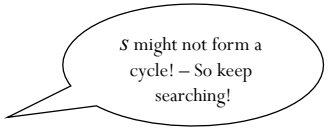


Heuristic Search for Liveness as Safety

(Jabbar & E., 2006)

- *Stage 1:* For a state $(s,s,0)$, perform a directed search for an accepting state s' in the never-claim.

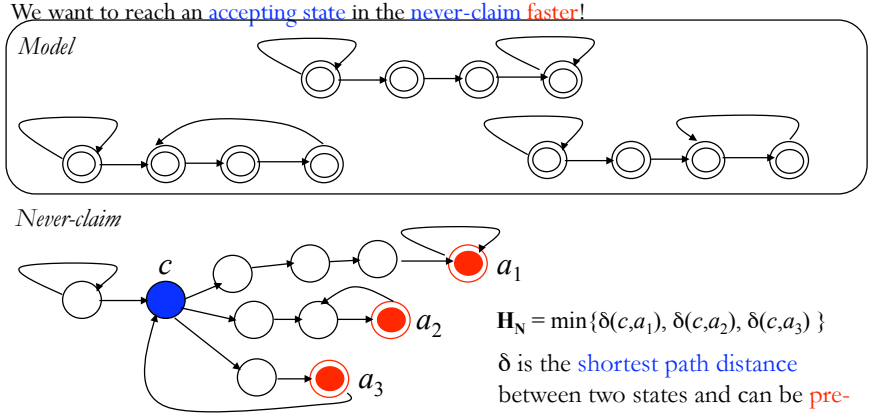
  When found
  - Spawn two children:
    - $(s, s, 1)$: Head of lasso found!
    - $(s, s, 0)$: Head of lasso not found!



- *Stage 2:* For a state $(s, s', 1)$, perform a directed search for s'.

Heuristic for the 1st stage – Head of the Lasso

- We want to reach an accepting state in the never-claim faster!

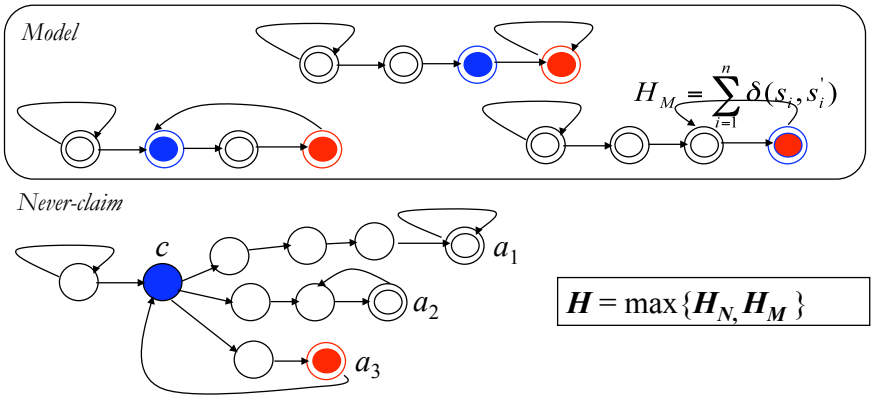


$\mathbf{H_N} = \min\{\delta(c,a_1), \delta(c,a_2), \delta(c,a_3)\}$

δ is the shortest path distance between two states and can be pre-computed.

Heuristics for the 2nd stage – Close the Lasso

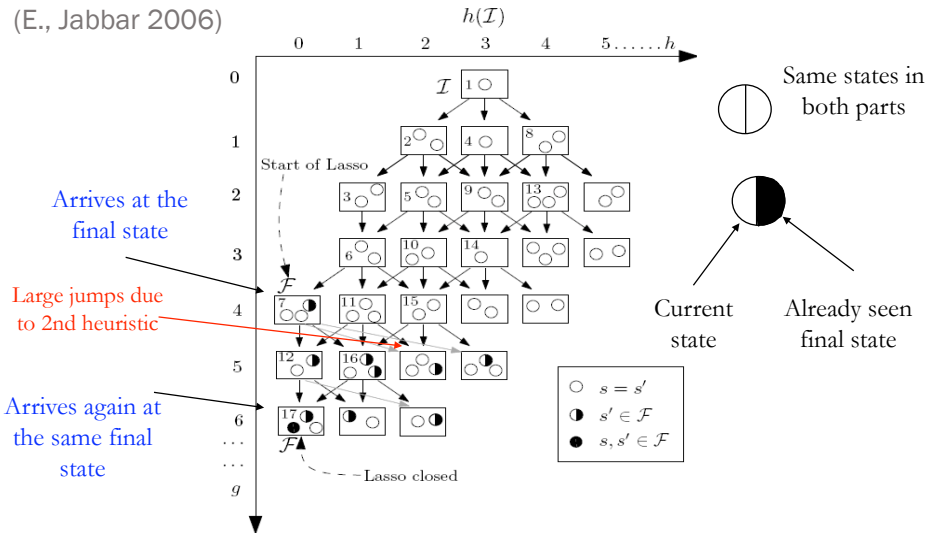
- We want to reach a particular state (in red) in both the model and the never-claim from my current state (in blue).



$\boldsymbol{H} = \max\{\boldsymbol{H_N}, \boldsymbol{H_M}\}$

External Directed LTL Model Checking

(E., Jabbar 2006)



I/Os of External A* for Liveness

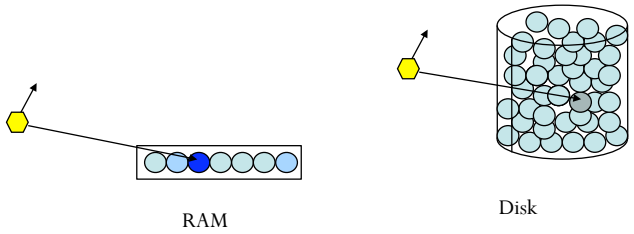
External memory algorithms are evaluated on the number of I/Os.

- Expansion: Linear I/O $O(Scan(|V| \times |F|))$
- Delayed Duplicate Detection:
  - Removing duplicates from the same buffer: $O(sort(|E| \times |F|))$
  - Subtracting previous levels: $O(l \times Scan(|V| \times |F|))$; where l is locality.

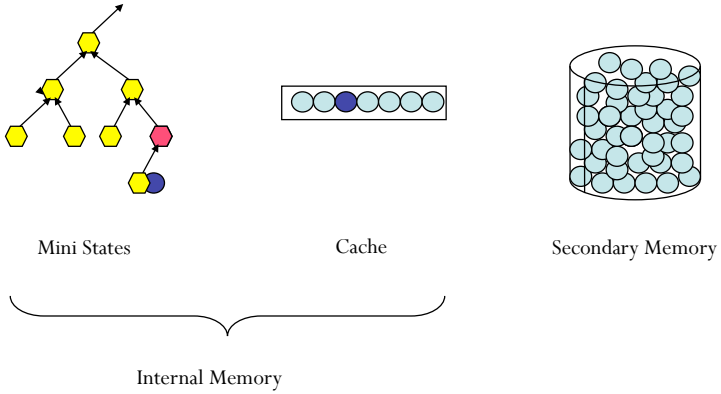
I/O Complexity = $O(sort(|E|\times|F|) + l \times Scan(|V|\times|F|))$

Alternative Approaces: Mini-States (E. et al. 2006)

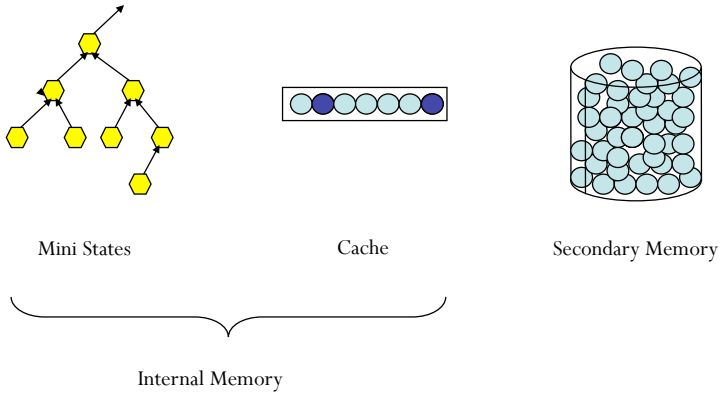
- Keep pointer to a state in RAM or on Disk
- Keep pointer to the predecessor mini state
- constant size



Expanding a State



Flushing the Cache



Alternative Approaches

- Dill et al. (1996) – Cache like mechanism to flush states.
- Kristensen and Mailund (2003) – Sweep line algorithm. Saves the closed list on disk in form of a spanning tree.
- Hammer and Weber (2006) – Hash-based partitioning of the state space. Utilizes compression and Bloom filters.
- Bao and Jones (2005) – Hash-based partitioning with each partition fitting into main memory.
- Barnat, Brim, Simecek (2007/08) Alternatives for External LTL – using STXXL
- Lamborn and Hansen (2008) - Layered Duplicate Detection
- Evangilista (2008) - Dynamic Delayed Duplicate Detection

Advanced Topics

- External Value Iteration
- Semi-External-Memory Graph Search
  - (Minimal) Perfect Hash Functions
  - c-bit Semi-Externality
- Flash Memory (Solid State Disk)
  - Immediate Duplicate Detection
  - Hashing with Fore- and Background Memory

Markov Decision Processes

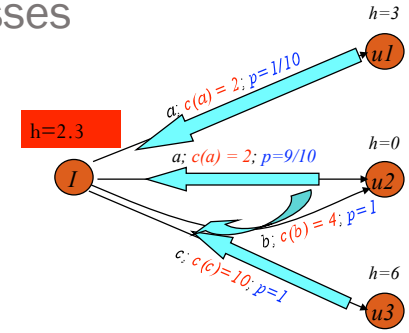
Given: Finite State-Transition System

Probabilistic + Non-deterministic

Action *a*: 2 + 1/10 x 3 + 9/10 x 0 = 2.3

b: 4 + 1 x 0 = 4

c: 10 + 1 x 6 = 16



$$h(u) = \begin{cases} c_{\mathcal{T}}(u) & \text{if } u \in \mathcal{T} \\ \min_{a \in A(u)} \left\{ c(a,u) + \sum_{v \in \Gamma(u,a)} P_a(v \mid u) \cdot h(v) \right\} & \text{otherwise} \end{cases}$$

Find: Optimal h-value assignment

Uniform Search Model:

Deterministic

$$h(u) = \begin{cases} 0 & \text{if } u \in \mathcal{T} \\ \min_{v \in \Gamma(u)} \{ c(a,u) + h(v) \} & \text{otherwise} \end{cases}$$

Non-Deterministic

$$h(u) = \begin{cases} 0 & \text{if } u \in \mathcal{T} \\ \min_{a \in A(u)} \left\{ c(a,u) + \sum_{v \in \Gamma(u,a)} h(v) \right\} & \text{otherwise} \end{cases}$$

Probabilistic

$$h(u) = \begin{cases} c_{\mathcal{T}}(u) & \text{if } u \in \mathcal{T} \\ \min_{a \in A(u)} \left\{ c(a,u) + \sum_{v \in \Gamma(u,a)} P_a(v \mid u) \cdot h(v) \right\} & \text{otherwise} \end{cases}$$

Internal Memory Value Iteration

Algorithm 1 Value Iteration

Input: State space $\mathcal{S}$; initial value func. h; tolerance $\epsilon \geq 0$
Output: Optimal value function h^*

```
1:  for all u ∈ S do
2:      h_0(u) ← h(u)
3:  end for
4:  t ← 0; Res ← +∞
5:  while t < t_max ∧ Res > ε do
6:      Res ← 0
7:      for all u ∈ S do
8:          Apply update rule for h_{t+1}(u) based on the model.
9:          Res ← max{|h_{t+1}(u) − h_t(u)|, Res}
10:     end for
11:     t ← t + 1
12: end while
13: return h_{t−1}
```

ε-Optimal for solving MDPs, AND/OR trees…

Problem:
Needs to have the whole state space in the main memory.

External-Memory Algorithm for Value Iteration

- What makes value iteration different from the usual external-memory search algorithms?
- **Answer:**
  - Propagation of information from states to predecessors!

➔ Edges are more important than the states.

Ext-VI works on Edges:

$$(u, v, h(v), a), \text{ where } u \xrightarrow{a} v$$

External Memory Value Iteration

- **Phase I:** Generate the edge space by External BFS.
- *Open*(0) = *Init*; *i* = -1
- **while** (*Open*(*i*-1) != empty)
  - *Open*(*i*) = Succ(*Open*(*i*-1))
  - Externally-Sort-and-Remove-Duplicates(*Open*(*i*))
  - **for** *loc* = 1 **to** Locality(*Graph*)
    - *Open*(*i*) = *Open*(*i*) \ *Open*(*i* - *loc*)

  } *Remove previous layers*
  - *i*++
- **endwhile**

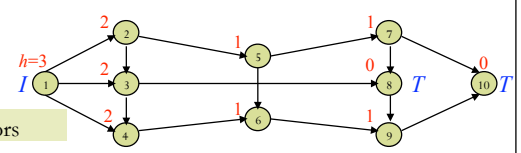
Merge all BFS layers into one edge list on disk!

$Open_t = Open(0) \cup Open(1) \cup \ldots \cup Open(\text{DIAM})$

$Temp = Open_t$

Sort $Open_t$ wrt. the successors; Sort *Temp* wrt. the predecessors

Working of Ext-VI *Phase-II*



Temp : Edge List on Disk – Sorted on Predecessors

h= 3 2 2 2 2 1 2 0 1 1 1 1 0 0 0 0

{(Ø, 1), (1,2), (1,3), (1,4), (2,3), (2,5), (3,4), (3,8), (4,6), (5,6), (5,7), (6,9), (7,8), (7,10), (9,8), (9,10)}

$h'(u) = 1 + \min_{\forall v \in Succ(u)} \{h(v)\}$

{(Ø,1), (1,2), (1,3), (2,3), (1,4), (3,4), (2,5), (4,6), (5,6), (5,7), (3,8), (7,8), (9,8), (6,9), (7,10), (9,10)}

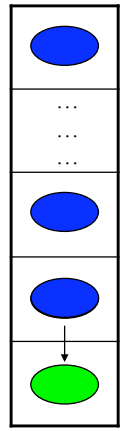
h= 3 2 2 2 2 2 1 1 1 1 0 0 0 1 0 0

h'= 3 2 1 1 2 2 2 2 2 1 0 0 0 1 0 0

Open$_t$: Edge List on Disk – Sorted on Successors

Alternate sorting and update until residual < epsilon

Complexity Analysis



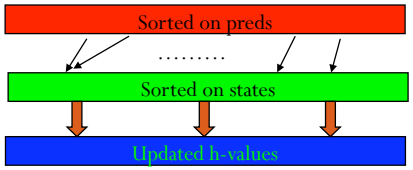
- **Phase-I: External Memory Breadth-First Search.**
- Expansion:
  - Scanning the red bucket: $O(scan(|E|))$
- Duplicates Removal:
  - Sorting the green bucket having one state for every edge from the red bucket.
  - Scanning and compaction: $O(sort(|E|))$
- Subtraction:
  - Removing states of blue buckets (duplicates free) from the green one: $O(l \times scan(|E|))$

Complexity of Phase-I:

$O(l \times scan(|E|) + sort(|E|))$ I/Os

Complexity Analysis

- **Phase-II: Backward Update**
- Update:
  - Simple block-wise scanning.
  - Scanning time for red and green files: $O(scan(|E|))$ I/Os
- External Sort:
  - Sorting the blue file with the updated values to be used as red file later: $O(sort(|E|))$ I/Os



Total Complexity of Phase-II:
For t_{max} iterations,

$O(t_{max} \times sort(|E|))$ I/Os

Semi-External EM Search[E., Sanders, Simecek 2008]

- generate state space with external BFS
- construct perfect hash function (MPHF) from disk
- use bit-state hash table Visited[h(u)] in RAM and stack on disk to perform cycle detection DFS

➔I/Os for Ex-BFS + const. scans and sorts

Optimal counter-examples:

➔I/Os for Ex-BFS + |F| scans

On-the-fly by iterative deepening (bounded MC)

➔I/Os for Ex-BFS + max-BFS-depth scans

Semi-Externality

Graph search algorithm A is c-bit semi-external if for each implicit graph G = (V,E) RAM requirements are at most O(vmax) + c· |V| bits.

O(vmax) covers the RAM needed for program code, auxiliary variables, and storage of a constant amount of vertices.

Lower bound

$\log\log|U| + (\log|E|)|V| + O(\log|V|)$ bits

Reduction in Practice

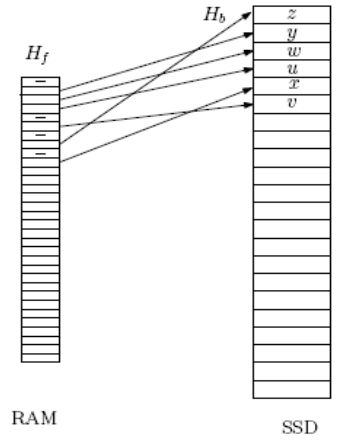
Model	Number of Vertices	v_{max}	ϵ_s	MPHF Size (bits/vertex)
Elev.2(16),P4	173,916,122	30 bytes	94	4.941
Lamport(5),P4	74,413,141	24 bytes	99	4.941
MCS(5),P4	119,663,657	28 bytes	91	4.941
Peterson(5),P4	284,942,015	32 bytes	177	4.941
Phils(16,1),P3	61,230,206	50 bytes	47	4.941
Ret.(16,8,4),P2	31,087,573	91 bytes	553	4.941
Szyman.(5),P4	419,183,762	32 bytes	223	4.941

Flash-Memory Graph Search

[E., Sulewski, 2008, Barnat, Brim, E., Simecek, Sulewski 2008]

- Solid State Disk operate as trade-off between RAM and Hard Disk
- On NAND technology, random reads are fast, random writes are slow
- With refined hashing, immediate duplicate detection becomes feasible for external memory graph search (CPU usage > 70%)
- Beats DDD in large search depth ….

Compression Strategy



Conclusion

- Disk-based algorithms with I/O complexity analysis.
- Can pause-and-resume execution to add more hard disks.
  - Error trace:
    - No predecessor pointers!
    - Save the predecessor with each state.
    - Trace back from the goal state to the start state breadth-wise.
  - Disk space eaten by duplicate states:
    - Start "Early" Delayed Duplicate Detection

Applications & Future Extensions

Applications:

- Sequence Alignment Problem
- Parallel External C++ Model Checking

In Implementation:

- Partial-Order Reduction
- Pipelined I/Os – keep block in the memory as long as possible